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Exact wavefunctions for a time-dependent Coulomb potential

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Abstract

The one-dimensional Schrödinger equation associated with a time-dependent Coulomb potential is studied. The invariant operator method (Lewis and Riesenfeld) and unitary transformation approach are employed to derive quantum solutions of the system. We obtain an ordinary second-order differential equation whose analytical exact solution has been unknown. It is confirmed that the form of this equation is similar to the radial Schrödinger equation for the hydrogen atom in a (arbitrary) strong magnetic field. The qualitative properties for the eigenstates spectrum are described separately for the different values of the parameter ω_0 appearing in the x^2 term, x being the position, i.e., $\omega_0 > 0$, $\omega_0 < 0$ and $\omega_0 = 0$. For the $\omega_0 = 0$ case, the eigenvalue equation of invariant operator reduces to a solvable form and, consequently, we have provided exact eigenstates of the time-dependent Hamiltonian system.

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1. Introduction

The problem of dynamical system described by time-dependent Hamiltonian has drawn special interest from theoretical physicists for a long time. This has happened due to its apparent applicability for understanding diverse problems in different areas of physics. For example, the construction of a time-dependent potential is necessary in studying the collisions of fast ions with metal surfaces [1], the charged-particle beams in accelerators [2] and the electronic structure of atoms in a time-dependent field [3].

A great deal of attention has been paid to some specific problems of time-dependent oscillators, for instance, the time-dependent singular oscillator and the oscillator driven by arbitrary force. In fact, these specific problems have been studied extensively in different

directions by many authors who have obtained its explicit closed-form solutions [4–22]. The construction of the invariants (constants of motion) which describe a quantum system governed by a time-dependent Hamiltonian has attracted much attention in the literature (see [5] and references therein). Lewis and Riesenfeld [23] have shown that, if the system admits an invariant $I(t)$, it is possible to find a privileged basis associated with the eigenstates of this invariant. The complete wavefunctions which evolve according to the time-dependent Schrödinger equation can be obtained by multiplying a suitable time-dependent phase factor by the eigenstates of $I(t)$.

In the meantime, the problem of the hydrogen atom still receives considerable interest with applications in various areas of both theoretical and experimental physics (see, for example, [24]). For a suitable description of the quantum feature for the hydrogen atom, it is necessary to take into account the Coulomb interaction which is a long-range interaction that can be efficiently managed. The investigation of the problem for Coulomb potential has its own relevance, since one can study bound states, degeneracy issues, singularity of the potential and the corresponding eigenfunctions.

On the other hand, the Schrödinger equation for the 2D hydrogen atom which has been considered in [25] under the influence of the strong magnetic field is integrable and even separable so that the radial equation effectively reduces to a one-dimensional problem. Nevertheless, it is well known that its radial Schrödinger equation cannot be solved exactly. The authors of [25] have described the qualitative properties of the energy spectrum and proposed a semianalytical method in order to numerically calculate the eigenenergies. However, the representation matrix of the Hamiltonian in the Landau basis is calculated analytically and, of course, is exactly known.

In this paper, we try to approach the general solutions of the one-dimensional time-dependent Schrödinger equation for the charged particle moving under the action of central forces which are associated with the time-dependent Coulomb potential $V(x, t)$ where x is a distance. For the motion of electrons in the hydrogen atom, the central-force field can be specified by the Coulomb potential $V(x, t) = -Z(t)/x$, where $Z(t)$ is a parameter determined by the charges of the electron and the nucleus and the dielectric constant. Some researchers have considered a time-dependent dielectric constant [26–30] which is evidently responsible for the time dependence of the Coulomb potential. The problem of a single particle which has a time-dependent mass, moving around a center of time-dependent Coulomb force, may also be a typical example associated with the time-dependent Coulomb potential. Dodonov *et al* studied the quantum problem of the time-dependent Coulomb potential on the basis of integral of motion and, as a result, obtained exact propagators associated with the system [31]. As far as we know, the time-dependent Coulomb potential can be employed in several dynamical systems. Soff calculated spin polarization of electrons induced by the strong magnetic field which is created by collisions of very heavy ions, and subjected to instantaneous Coulomb potential which explicitly depends on time according to the time dependence of heavy ion charge density [32]. Staudt and Keitel studied the ionization behavior of helium under the influence of the strong laser field in the high-frequency regime by introducing the time-dependent repulsive Coulomb potential described by mean-field ansatz [33]. The effect of time-dependent screening of a negative charge has been investigated by introducing time-dependent effective Coulomb potential [34].

The aim of our present work is to investigate the exact wavefunctions of quantum system associated with the time-dependent Coulomb potential using the Lewis and Riesenfeld invariant method. The one-dimensional equation obtained for this system is similar to the radial Schrödinger equation of the hydrogen atom. Our motivation stems from the fact that the

exact analytical solutions of the time-independent radial Schrödinger equation of the hydrogen atom, which is an ordinary second-order differential equation, are unknown yet.

This paper is organized as follows. In section 2, we consider a Hamiltonian which describes a time-dependent Coulomb potential system and construct a corresponding invariant operator. The eigenfunctions of the invariant operator and exact wavefunctions which satisfy the Schrödinger equation are investigated in section 3. In section 4, we applied our development to two special cases in order to promote our understanding of the theory. The concluding remarks are placed in section 5.

2. Hamiltonian and invariant operator

Let us recall the general method to introduce invariant operators [23] for a system specified by a time-dependent Hamiltonian $H(t)$ and a corresponding evolution operator $\mathcal{U}(t)$. An invariant $I(t)$ may be constructed by taking advantage of the fact that its time-derivative results in zero:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0. \quad (1)$$

It possesses a remarkable property that any eigenstate of $I(0)$ evolves into an eigenstate of $I(t)$. Then, if the set of reference eigenstates $\{\phi_n(t)\}$ for the operator $I(t)$ is continuous with respect to t (all eigenstates are associated with the same time-independent eigenvalue ε_n), the corresponding global phases $\theta_n(t)$ are defined by the relation associated with the wavefunctions $\psi_n(t)$:

$$\psi_n(t) = \mathcal{U}(t)\phi_n(0) = e^{i\theta_n(t)}\phi_n(t). \quad (2)$$

Of course, $\psi_n(t)$ follow the Schrödinger equation of the form

$$i\hbar \frac{\partial}{\partial t} \psi_n(t) = H(t)\psi_n(t), \quad (3)$$

and $\theta_n(t)$ satisfies the relation

$$\hbar \frac{d}{dt} \theta_n(t) = \langle \phi_n(t) | i\hbar \frac{\partial}{\partial t} - H | \phi_n(t) \rangle. \quad (4)$$

Our task, in this section, is to derive the invariant operator of the system governed by the Hamiltonian:

$$H(x, p, t) = A(t)p^2 + C(t) \left(\frac{1}{x}p + p\frac{1}{x} \right) + \frac{E(t)}{x^2} - \frac{Z(t)}{x}, \quad (5)$$

which is defined in the half space $x \geq 0$, where $A(t)$, $C(t)$, $E(t)$ and $Z(t)$ are time-dependent coefficients with $Z(t) > 0$. $E(t)$ and $Z(t)$ represent the strength of the singular potentials. The time variation of such strengths was investigated by Dent and Fairbairn [35]. This Hamiltonian is exactly the one related to the hydrogen atom and the term $(1/x)p + p(1/x)$ gives the expression containing $\frac{1}{x} \frac{\partial}{\partial x}$ in coordinate space.

Now, let us find the exact invariant starting from

$$I(x, p, t) = \alpha(t)x^2 + \gamma(t)p^2 + \beta(t)(xp + px) + \delta(t) \left(\frac{1}{x}p + p\frac{1}{x} \right) + \frac{\lambda(t)}{x^2} - \frac{\eta(t)}{x} + \xi(t), \quad (6)$$

where $\alpha(t) - \xi(t)$ are time-dependent coefficients which should be determined afterward.

Substitution of equations (5) and (6) into the Liouville–von Neumann equation represented in equation (1) gives the equations for the coefficients:

$$\dot{\alpha} = 0, \quad (7)$$

$$\dot{\beta} = -2\alpha A, \tag{8}$$

$$\dot{\gamma} = -4\beta A, \tag{9}$$

$$\dot{\eta} = -2\beta Z, \tag{10}$$

$$\dot{\delta} = -4\beta C, \tag{11}$$

$$\dot{\lambda} = -4\beta E, \tag{12}$$

$$\dot{\xi} = -4\alpha C, \tag{13}$$

$$\gamma Z = \eta A, \tag{14}$$

$$\delta Z = \eta C, \tag{15}$$

$$\gamma C = \delta A, \tag{16}$$

$$\lambda A = \gamma E, \tag{17}$$

$$\lambda C = \delta E. \tag{18}$$

The ratios $\frac{C(t)}{A(t)}$ and $\frac{E(t)}{A(t)}$ are constants. For convenience, let us call them k_1 and k_2 , respectively. This fact can be easily inferred from equations (7)–(17). Note that this can also be explained by making use of the total energy (5) which is a function of canonical variables x and p . If we define a generalized kinetic momentum as $P = \frac{1}{2A}(\frac{d}{dt}x)$, it may be possible to eliminate the canonical momentum p from equation (5) using $P = \frac{1}{2A}(\frac{d}{dt}x) = p + k_1/x$ which is similar to the canonical momentum for a radial 2D hydrogen atom. Through this process, equation (5) becomes just the sum of a kinetic energy and the Coulomb electrostatic potential energy, namely $H(x, p, t) = \frac{1}{4A}(\frac{d}{dt}x)^2 - \frac{Z(t)}{x}$ (note that we have taken as $k_2 = k_1^2$).

Now we can solve equations (7)–(18) to give the explicit values of the coefficients in the invariant operator. Thus we have

$$\alpha(t) = \alpha_0, \tag{19}$$

$$\beta(t) = \beta_0 - 2\alpha_0 \int_0^t A(t') dt', \tag{20}$$

$$\gamma(t) = \gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2, \tag{21}$$

$$\delta(t) = \frac{\delta_0}{\gamma_0} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right), \tag{22}$$

$$\eta(t) = \frac{\eta_0}{\gamma_0^{\frac{1}{2}}} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right)^{\frac{1}{2}}, \tag{23}$$

$$\lambda(t) = \frac{\lambda_0}{\gamma_0} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right), \tag{24}$$

$$\xi(t) = \frac{2\delta_0}{\gamma_0} \left(\beta_0 - 2\alpha_0 \int_0^t A(t') dt' \right), \tag{25}$$

under the proper choice of some integral constants, $\delta_0 = k_1\gamma_0$ and $\lambda_0 = k_2\gamma_0$. By inserting equations (19)–(25) into (6), we obtain the expression for the invariant operator:

$$\begin{aligned}
 I(x, p, t) = & \alpha_0 x^2 + \left(\beta_0 - 2\alpha_0 \int_0^t A(t') dt' \right) (xp + px) \\
 & + \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right) p^2 \\
 & + \frac{\delta_0}{\gamma_0} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right) \left(\frac{1}{x} p + p \frac{1}{x} \right) \\
 & - \frac{\eta_0}{\gamma_0^{\frac{1}{2}}} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right)^{\frac{1}{2}} \frac{1}{x} \\
 & + \frac{\lambda_0}{\gamma_0} \left(\gamma_0 - 4\beta_0 \int_0^t A(t') dt' + 4\alpha_0 \left[\int_0^t A(t') dt' \right]^2 \right) \frac{1}{x^2} \\
 & + \frac{2\delta_0}{\gamma_0} \left(\beta_0 - 2\alpha_0 \int_0^t A(t') dt' \right). \tag{26}
 \end{aligned}$$

3. Exact quantum solutions

The key point of our analysis is to perform the time-dependent unitary transformation such that

$$\Phi_n(x) = U(t)\phi_n(x, t), \tag{27}$$

where a time-dependent unitary operator $U(t)$ is given by

$$U(t) = V(t)\Lambda(t) = \exp\left(\frac{i\beta(t)}{2\hbar\gamma_0}x^2\right) \times \exp\left(\frac{i}{2\hbar} \ln\left(\frac{\gamma(t)}{\gamma_0}\right)^{\frac{1}{2}}(xp + px)\right). \tag{28}$$

It can be easily shown that, under this transformation, the coordinate and momentum operators change according to

$$x \longrightarrow x = U(t)xU(t)^{-1} = \left(\frac{\gamma(t)}{\gamma_0}\right)^{\frac{1}{2}}x, \tag{29}$$

$$p \longrightarrow p = U(t)pU(t)^{-1} = \left(\frac{\gamma(t)}{\gamma_0}\right)^{-\frac{1}{2}}\left(p - \frac{\beta(t)}{\gamma_0}x\right). \tag{30}$$

Hence, the operator I changes into time-independent operator $I_0 = UIU^{-1}$. In other words, the invariant operator (26) becomes

$$I(t) \longrightarrow I_0 = UIU^{-1} = \gamma_0 p^2 + (\alpha_0\gamma_0 - \beta_0^2)x^2 + \delta_0\left(\frac{1}{x}p + p\frac{1}{x}\right) + \frac{\lambda_0}{x^2} - \frac{\eta_0}{x}. \tag{31}$$

Thus, the eigenvalue equation for the transformed invariant operator can be simply represented in the form

$$\left(\gamma_0 p^2 + (\alpha_0\gamma_0 - \beta_0^2)x^2 + \delta_0\left(\frac{1}{x}p + p\frac{1}{x}\right) + \frac{\lambda_0}{x^2} - \frac{\eta_0}{x}\right)\Phi_n(x) = \varepsilon_n\Phi_n(x). \tag{32}$$

This is the fundamental one-dimensional ordinary second-order differential equation, analogous to that for the radial wavefunction Φ_n as a function of the dimensionless polar radius x , which we are going to study.

According to values of $\omega_0 = (\alpha_0\gamma_0 - \beta_0^2)$, i.e. positive, negative or zero, we can distinguish three cases.

- (i) *Case 1:* $\omega_0 > 0$. In this case, the system becomes the same as the radial Schrödinger equation of the hydrogen atom in the (arbitrary) strong magnetic field, which is an ordinary second-order differential equation whose analytical exact solution is unknown. The problem of this situation has been treated by Robnik and Romanovsky [25]. They described the qualitative properties of the energy spectrum and employed a semianalytical method in order to calculate the numerical eigenenergies. To estimate the ground-state energy and the higher-order excited ones, they also used a number of useful analytical approximation methods such as the semiclassical approximation, the perturbation method, the variational method and the Taylor power expansion of the potential around the minimum.
- (ii) *Case 2:* $\omega_0 < 0$. If we introduce a positive notation ϖ such that $\omega_0 = -\varpi^2 = (i\varpi)^2$, equation (32) becomes

$$\left(\gamma_0 p^2 - \varpi^2 x^2 + \delta_0 \left(\frac{1}{x} p + p \frac{1}{x} \right) + \frac{\lambda_0}{x^2} - \frac{\eta_0}{x} \right) \Phi_n(x) = \varepsilon_n \Phi_n(x). \quad (33)$$

It is clear that this equation is similar to the so-called inverted radial Schrödinger equation of the hydrogen atom in the (arbitrary) strong magnetic field. However, it cannot be solved exactly. Nevertheless, by applying the unitary transformation with $\Lambda_{\frac{\pi}{4}} = \exp\left(i\frac{\pi}{\hbar}(xp + px)\right)$, the situation convert to that of the $\omega_0 > 0$ case which requires much analytical work.

- (iii) *Case 3:* $\omega_0 = 0$. In this case, the invariant can be written as

$$I_0 = \gamma_0 p^2 + \delta_0 \left(\frac{1}{x} p + p \frac{1}{x} \right) + \frac{\lambda_0}{x^2} - \frac{\eta_0}{x}. \quad (34)$$

One may see that this is really equivalent to or reduces to that of the situation associated with a time-independent Hamiltonian. Therefore, the instantaneous eigenstates of the time-dependent system can be readily transformed to the eigenstates of the Hamiltonian related to the time-independent system. Then the eigenvalue equation (32) becomes

$$\left(\frac{\partial^2}{\partial x^2} + a \frac{1}{x} \frac{\partial}{\partial x} + \frac{\eta_0}{\hbar^2 \gamma_0 x} - b \frac{1}{x^2} + \frac{\varepsilon_n}{\hbar^2 \gamma_0} \right) \Phi_n(x) = 0, \quad (35)$$

where $a = \frac{2i\delta_0}{\hbar\gamma_0}$ and $b = \frac{i\delta_0}{\hbar\gamma_0} + \frac{\lambda_0}{\hbar^2\gamma_0}$.

At this stage, let us suppose that the spectrum of eigenvalues is discrete, i.e., the system is bound. This requirement can be met by putting $\varepsilon_n < 0$. To solve equation (35), we express Φ_n in the form

$$\Phi_n(x) = x^r e^{qx} \chi_n(x), \quad (36)$$

where

$$r = \frac{1}{2} - \frac{i\delta_0}{\hbar\gamma_0} + \sqrt{\frac{1}{4} - \frac{\delta_0^2}{\hbar^2\gamma_0^2} + \frac{\lambda_0}{\hbar^2\gamma_0}}, \quad (37)$$

$$q = \frac{1}{\hbar\gamma_0^{\frac{1}{2}}} \sqrt{-\varepsilon_n}. \quad (38)$$

By inserting equation (36) into (35) and after some rearrangement, we can obtain the differential equation

$$y \frac{\partial^2 \chi(y)}{\partial y^2} + (1+l-y) \frac{\partial \chi(y)}{\partial y} + \frac{1}{2} \left(\frac{-\eta_0}{\hbar^2 \gamma_0 q} - l - 1 \right) \chi(y) = 0, \quad (39)$$

where

$$y = -2qx, \tag{40}$$

$$l = \sqrt{1 - \frac{\delta_0^2}{\hbar^2 \gamma_0^2} + \frac{\lambda_0}{\hbar^2 \gamma_0}}. \tag{41}$$

In equation (39), $\chi_n(y)$ satisfies the associated Laguerre polynomial:

$$\chi_n(y) = L_n^l(y), \tag{42}$$

where

$$n = \frac{1}{2} \left(\frac{-\eta_0}{\hbar \gamma_0^{\frac{1}{2}} \sqrt{-\varepsilon_n}} - l - 1 \right). \tag{43}$$

Consequently, the constant eigenvalues ε_n are exactly given as

$$\varepsilon_n = \frac{-\eta_0^2}{\hbar^2 \gamma_0 (2n + l + 1)^2}. \tag{44}$$

The eigenfunctions of the invariant I_0 can, then, be represented in the form

$$\begin{aligned} \Phi_n(x) = & \left[(2n + l + 1)^{l+3} \frac{\Gamma(n + l + 1)}{\Gamma(n + 1)} \left(\frac{\hbar^2 \gamma_0}{2\eta_0} \right)^{l+2} \right]^{-\frac{1}{2}} x^{\frac{l}{2} + \frac{1}{2} - \frac{i\delta_0}{\hbar \gamma_0}} \\ & \times \exp\left(\frac{-\eta_0}{\hbar^2 \gamma_0 (2n + l + 1)} x \right) \times L_n^l \left(\frac{2\eta_0}{\hbar^2 \gamma_0 (2n + l + 1)} x \right). \end{aligned} \tag{45}$$

The complete normalized states for $I(t)$ are thus evaluated to be

$$\begin{aligned} \phi_n(x, t) = & U^{-1} \Phi_n(x) = \Lambda^{-1} V^{-1} \Phi_n(x) \\ = & \left[(2n + l + 1)^{l+3} \frac{\Gamma(n + l + 1)}{\Gamma(n + 1)} \left(\frac{\hbar^2 \gamma(t)}{2\eta(t)} \right)^{l+2} \right]^{-\frac{1}{2}} \\ & \times \left(\frac{\gamma(t)}{\gamma_0} \right)^{\frac{i\delta_0}{2\hbar \gamma_0}} x^{\frac{l}{2} + \frac{1}{2} - \frac{i\delta_0}{\hbar \gamma_0}} \exp\left(\frac{i\beta(t)}{2\hbar \gamma(t)} x^2 \right) \\ & \times \exp\left(\frac{-\eta(t)}{\hbar^2 \gamma(t) (2n + l + 1)} x \right) \times L_n^l \left(\frac{2\eta(t)}{\hbar^2 \gamma(t) (2n + l + 1)} x \right). \end{aligned} \tag{46}$$

There still remains the problem of finding the phases $\theta_n(t)$ which satisfy equation (4). Carrying out the unitary transformation by means of $U(t)$, equation (4) becomes

$$\hbar \frac{d}{dt} \theta_n(t) = \langle \Phi_n(x) | - \frac{A(t)}{\gamma(t)} I_0 - \frac{\delta_0 \dot{\gamma}(t)}{2\gamma_0 \gamma(t)} | \Phi_n(x) \rangle. \tag{47}$$

Then, with the help of equation (44), this equation can be easily evaluated so that we obtain the phases in the form

$$\theta_n(t) = \frac{\eta_0^2}{\hbar^3 \gamma_0 (2n + l + 1)^2} \int_0^t \frac{A(t')}{\gamma(t')} dt' - i \ln \left(\frac{\gamma(t)}{\gamma_0} \right)^{-\frac{i\delta_0}{2\hbar \gamma_0}}. \tag{48}$$

Therefore, by substituting equations (46) and (48) into (2), the exact n th-order solution of the Schrödinger equation (1) associated with the Hamiltonian $H(x, p, t)$ is

$$\begin{aligned} \psi_n(x, t) = & \left[(2n+l+1)^{l+3} \frac{\Gamma(n+l+1)}{\Gamma(n+1)} \left(\frac{\hbar^2 \gamma(t)}{2\eta(t)} \right)^{l+2} \right]^{-\frac{1}{2}} x^{\frac{l}{2} + \frac{1}{2} - \frac{i\eta_0}{\hbar\gamma_0}} \\ & \times \exp\left(\frac{i\beta(t)}{2\hbar\gamma(t)} x^2\right) \times \exp\left(\frac{-\eta(t)}{\hbar^2 \gamma(t)(2n+l+1)} x\right) \\ & \times \exp\left(\frac{i\eta_0^2}{\hbar^3 \gamma_0(2n+l+1)^2} \int_0^t \frac{A(t')}{\gamma(t')} dt'\right) \\ & \times L_n^l\left(\frac{2\eta(t)}{\hbar^2 \gamma(t)(2n+l+1)} x\right). \end{aligned} \quad (49)$$

4. Applications

In this section, we apply our developments to specific cases in order to promote our further understanding.

4.1. Hydrogenoid atom

Indeed, our study for the quantum solutions of the time-dependent Schrödinger equation can be applied to the problem of charged particle moving under the action of central forces associated with the time-dependent Coulomb potential. If we take $A(t) = (1/2m_e)$ and $Z(t) = (e^2/4\pi\epsilon_0)$ our wavefunctions $\psi_n(x, t)$ become analogous to those of the radial equation for the hydrogen atom in stationary case [24]. On the other hand, in case that $A(t) = (1/2\mu)$ and $Z(t) = (Ze^2/4\pi\epsilon_0)$, our problem is equivalent to that of a time-independent hydrogenoid atom.

4.2. One dimensional time-dependent Coulomb potential

For the simple case where $C(t) = 0$ and $E(t) = 0$, the system is equivalent to the one-dimensional Coulomb potential problem for the positive half region $x > 0$. Because of the singularity at the origin, solutions must be obtained separately for the two regions $x > 0$ and $x < 0$ and then appropriately matched at $x = 0$. Since $A(t)p^2 - Z(t)/|x|$ is symmetric for both sides of origin, the solutions for $x > 0$ can be extended to $x < 0$ so that we obtain even and odd wavefunctions. Thus, the corresponding *regular* solution is given in the form:

$$\begin{aligned} \psi_n(x, t) = & \left[(2n+2)^4 \frac{\Gamma(n+2)}{\Gamma(n+1)} \left(\frac{\hbar^2 \gamma(t)}{2\eta(t)} \right)^3 \right]^{-\frac{1}{2}} x \exp\left(\frac{i\beta(t)}{2\hbar\gamma(t)} |x|^2\right) \\ & \times \exp\left(\frac{-\eta(t)}{\hbar^2 \gamma(t)(2n+2)} |x|\right) \exp\left(\frac{i\eta_0^2}{\hbar^3 \gamma_0(2n+2)^2} \int_0^t \frac{A(t')}{\gamma(t')} dt'\right) \\ & \times L_n^1\left(\frac{2\eta(t)}{\hbar^2 \gamma(t)(2n+2)} |x|\right). \end{aligned} \quad (50)$$

If we take $A = \text{const}$ and $Z = \text{const}$, expression (50) is equivalent to the regular solution of the stationary Schrödinger equation for a one-dimensional Coulomb potential problem [36–40].

4.3. Coulomb potential with time-dependent mass

The developments of the present work can also be applied to the problem of Coulomb potential with time-dependent mass. The Hamiltonian system with time-dependent mass has wide applications and sometimes plays significant roles in diverse branches of physics. For instance, Colegarve and Abdalla employed the problem of an harmonic oscillator with time-dependent mass to study the problem of a Febry–Perot cavity which interacts with a heat reservoir [41]. Mandal has investigated nonclassical behavior of coherent light coupled to the oscillator, such as photon-bunching, photon-antibunching and nonclassical photon statistics, by introducing the quantum-driven oscillator model of time-dependent mass (and frequency) [42]. In particular, we take an exponentially decaying mass of the form $m(t) = m_0 e^{-\kappa t}$ where m_0 and κ are real constants. Besides, we choose time-dependent coefficients such as $A(t) = 1/(2m(t))$, $C(t) = C_0 e^{\kappa t}$, $E(t) = E_0 e^{\kappa t}$ and $Z(t) = km(t)$ where C_0 , E_0 and k are real constants. Then, the Hamiltonian in equation (5) can be written in the form

$$H = -\frac{\hbar^2}{2m_0 e^{-\kappa t}} \frac{\partial^2}{\partial x^2} - 2iC_0 \hbar e^{\kappa t} \frac{1}{x} \frac{\partial}{\partial x} + (E_0 + iC_0 \hbar) e^{\kappa t} \frac{1}{x^2} - e^{-\kappa t} \frac{m_0 k}{x}. \quad (51)$$

In this case, it is possible to integrate $\int_0^t A(t') dt'$, appearing in equations (20)–(25), so as to lead the corresponding wavefunctions in the form

$$\begin{aligned} \psi_n(x, t) = & \left[(2n+l+1)^{l+3} \frac{\Gamma(n+l+1)}{\Gamma(n+1)} \left(\frac{\hbar^2 \Pi_1(t)}{2\eta_0} \right)^{l+2} \right]^{-\frac{1}{2}} x^{\frac{l}{2} + \frac{1}{2} - \frac{i\eta_0}{\hbar\gamma_0}} \\ & \times \exp\left(\frac{i\Pi_2(t)}{2\hbar} x^2\right) \times \exp\left(\frac{-\eta_0}{\hbar^2 \Pi_1(t)(2n+l+1)} x\right) \\ & \times \exp\left(\frac{i\eta_0^2}{2m_0 \hbar^3 (2n+l+1)^2} \int_0^t \frac{dt'}{\Pi_1^2(t') e^{-\kappa t'}}\right) \\ & \times L_n^l \left(\frac{2\eta_0}{\hbar^2 \Pi_1(t)(2n+l+1)} x \right), \end{aligned} \quad (52)$$

where

$$\Pi_1(t) = \left[\gamma_0 \left(\gamma_0 - \frac{2\beta_0}{m_0 \kappa} (e^{\kappa t} - 1) + \frac{\alpha_0}{m_0^2 \kappa^2} (e^{\kappa t} - 1)^2 \right) \right]^{1/2}, \quad (53)$$

$$\Pi_2(t) = \frac{\gamma_0}{\Pi_1^2} \left(\beta_0 - \frac{\alpha_0}{m_0 \kappa} (e^{\kappa t} - 1) \right). \quad (54)$$

For $t \rightarrow 0$, equations (53) and (54) reduce to $\Pi_1(t) = \gamma_0$ and $\Pi_2(t) = \beta_0/\gamma_0$ and, consequently, the result (equation (52)) becomes the same as that of the stationary case. Although we manage the system only in one spatial dimension, it will be evident that our techniques can also be employed to higher-dimensional models without radical modification [43–45].

5. Conclusion

We investigated Schrödinger solutions of a one-dimensional time-dependent Hamiltonian system involving time-dependent Coulomb potential. To do this we employed the invariant operator and unitary transformation methods together. The original invariant given in equation (6) with equations (19)–(25) is explicitly a function of t , though its time derivative

vanishes: $dI/dt = 0$. However, the invariant, equation (31), which is transformed by $U(t)$, has a simple form and is no longer a function of t . Due to this fact, the management of the transformed invariant in order to solve the eigenvalue equation is much better than treating the original one. We discussed the eigenvalue equation of the transformed invariant operator separately for the three cases, i.e., $\omega_0 > 0$, $\omega_0 < 0$ and $\omega_0 = 0$. As can be seen from equation (49), the Schrödinger solutions have been represented in terms of associated Laguerre function and are the same as that of the eigenstates of the invariant operator except for some time-dependent phase factors. Since we do not use any approximation or perturbation method, our results are exact provided that the explicit value of integration $\int_0^t A(t') dt'$ appearing in equations (20)–(25) is obtained.

Note that, the strength of Coulomb potential is controlled by the coefficient $Z(t)$ which acts as a coupling parameter. There is an interesting feature associated with the singular potential such as Coulomb interaction, which is known as the ‘Klauder phenomenon’ [46]. According to this phenomenon, once the perturbation $-Z(t)/x$ is switched on, it is impossible to turn off it thoroughly due to the occurrence of vestigial effects which make the interaction continue permanently. By suitable choices of time-dependent coefficients given in equation (5), our developments can be applied to various types of quantum systems that are described by a particular time-dependent Hamiltonian involving Coulomb potential. To promote our understanding, we applied our theory to three special systems namely, the hydrogenoid atom the one dimensional time-dependent Coulomb potential, and a Coulomb potential system with time-dependent mass.

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